

Inducing the Lovelock action

Jan-Peter Börnsen^{1*} and Anton E. M. van de Ven^{2†}

¹ II. Institute for Theoretical Physics, University Hamburg, Germany

² University College Utrecht, The Netherlands

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Abstract

We re-analyze a possible ambiguity in the application of dimensional regularization to Einstein-Gauss-Bonnet gravity, arising from the way one treats the Gauss-Bonnet term [1]. It is demonstrated that the addition of such a term to the action gives rise to a non-minimal graviton wave operator, but does not produce new on shell divergences at one loop order in $d = 4$. However, from a d -dimensional perspective the Gauss-Bonnet term is shown to generate new divergences in the form of the six-dimensional Euler density. The conjecture that one would next produce the eight-dimensional Euler term is shown to be false.

1 Introduction

The natural generalization of the Einstein-Hilbert action to a spacetime of dimension greater than four is provided by the Lovelock action [2, 3]; see e.g. [4] for a review. It consists of the sum of all dimensionally continued Euler densities, each term coming with its own coupling constant. The Lovelock action gives rise to a symmetric conserved field equation which contains derivatives of the metric no higher than second order and is quasi-linear in these second derivatives. This leads to a well-defined classical Cauchy problem [5, 6], and to unitarity at the quantum level, unlike generic higher derivative actions [7, 8]. Unitarity is also the reason why one expects the Lovelock action to make its appearance in the low-energy limit of string theory [9, 10]. More recently, numerous papers have appeared in which this same action appears in connection with brane scenarios in string theory, see e.g. [4, 11]. This partly motivated our examination of the ultraviolet behavior of the Lovelock action.

It is well known that a perturbative treatment of Einstein gravity gives rise to a one loop finite S-matrix in four dimensions. The background field method and dimensional regularization predict one loop divergences of curvature-squared form [12]. Although these do not have the functional form of the classical action, the vacuum field equations combined with the Gauss-Bonnet identity would imply that such divergences do not survive. However, this reasoning was criticized in a little known paper by Capper and Kimber [1]. They pointed out that dimensional regularization requires one to work consistently in d dimensions and, although the field equation can trivially be continued,

*E-mail: jan-peter.boernsen@desy.de

†E-mail: A.E.M.vandeVen@uu.nl

there does not exist a d -dimensional Gauss-Bonnet identity. In short, the $d \rightarrow 4$ limit of the expression

$$\frac{1}{d-4} \int d^d x \sqrt{g} (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2)$$

is ill defined. This motivated Capper and Kimber to add a Riemann-squared term to the classical action. They then demonstrated that this extra term has no effect on tree-level graviton-graviton scattering. This was shown to be not due to explicit factors of $d-4$, as none appear, but rather caused by the imposition of on shell conditions and setting d equal to four for all external legs. Although no one loop calculation was performed, an appeal to consistency with other regularization schemes led them to conclude that the one loop S-matrix of standard gravity is finite after all. The re-examination in the present paper answers the question of Capper and Kimber for the divergent parts of the effective action at one loop level. No changes are found, so we confirm what one would naively have expected from adding such a $d=4$ topological term. However, in the process we discover that from the d -dimensional point of view advocated by Capper and Kimber, adding the Gauss-Bonnet term to the classical action next requires the $d=6$ Euler density as a counter term. Hence, it would seem that the renormalization process induces the full Lovelock action.

Besides the well known general difficulty of any calculation in quantum gravity, a further reason why the problem was not yet studied beyond tree level is that one is then faced with a non-minimal wave operator. If the leading part of a wave operator equals the Laplacian, one speaks of a minimal operator. In gauge theories this form can usually be arranged by a judicious gauge choice. Minimal wave operators allow the use of the powerful Schwinger-DeWitt method and convenient background field algorithms are then available. We will show that the addition of the Gauss-Bonnet term inevitably leads to a non-minimal wave operator for the graviton. An elegant extension of the Schwinger-DeWitt method to the non-minimal case was given in [13] but we did not rely on this work. Since the offending term in the wave operator turns out to be of first order in the curvature, we may treat it perturbatively and thus return to the minimal setting. Using the fully covariant Schwinger-DeWitt method, we reproduce and extend to curved space an earlier flat space algorithm [14]. There, 't Hooft's background field algorithm [15] was generalized to second order in the non-minimal part of the wave operator (see also the recent systematic work of [16, 17]).

In related work, Berredo-Peixoto and Shapiro recently demonstrated that no new one loop divergences are generated upon adding the Gauss-Bonnet term to either conformal [18] or general [19] higher derivative gravity. Such theories are known to be renormalizable, though not unitary. The authors of [18, 19] employed dimensional regularization and the Schwinger-DeWitt method for their involved calculations. Interestingly, the Gauss-Bonnet term was shown to actually affect the d -dimensional renormalization group equations and new non-trivial fixed points were found. However, these papers do not answer the original issue of [1]: Although the classical action in [19] includes Einstein-Hilbert and cosmological terms, a higher derivative gauge fixing term was chosen which does not allow one to regain the special case of two-derivative gravity. Hence, the analysis in [19] excludes the case of Lovelock gravity we are interested in.

2 Perturbative expansion of the Lovelock action

In a d dimensional Riemann space, the Lovelock action is given by

$$S = \frac{-1}{16\pi G} \sum_{k=0}^{[d/2]} \lambda_k S_k \quad , \quad \lambda_1 \equiv 1 \quad , \quad S_k = \int dv \frac{(2k)!}{2^k} R^{\mu_1 \mu_2}_{\mu_1 \mu_2} \dots R^{\mu_{2k-1} \mu_{2k}}_{\mu_{2k-1} \mu_{2k}} \quad . \quad (1)$$

For compactness, the Riemann curvature tensor $R_{\mu\nu}{}^{\rho\sigma}$ has been written here as $R^{\rho\sigma}_{\mu\nu}$ and $dv = d^d x \sqrt{g}$. In $d = 2k$, the integrand of S_k is proportional to the Euler density. Note the total antisymmetry on $2k$ indices. For given dimension d , the Schouten identity implies that the maximum value of k is the integer part of $d/2$, denoted here as $[d/2]$. The λ_k are arbitrary coupling constants. In four dimensions this action equals

$$S = \frac{-1}{16\pi G} \int dv [\lambda_0 + R + \lambda_2 (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2)] \quad (2)$$

One recognizes here the cosmological constant, Einstein-Hilbert term, and four-dimensional Euler density, also known as Gauss-Bonnet term. The latter is only locally a total derivative and cannot be written as the divergence of a vector field. In higher dimensions, further dimensionally continued Euler densities appear. For later reference we note that in a Ricci flat space S_3 reduces to (C represents the Weyl tensor)

$$S_3 = \int dv C_{\mu}{}^{\nu}{}_{\alpha}{}^{\beta} C_{\nu}{}^{\rho}{}_{\beta}{}^{\gamma} (C_{\rho}{}^{\mu}{}_{\gamma}{}^{\alpha} - 2C_{\rho}{}^{\alpha}{}_{\gamma}{}^{\mu}) \equiv - \int dv (C_W^3 - 2C_M^3) \quad (3)$$

Here, the labels W and M stand for Wheel and Möbius, respectively. Using the graphical notation of the Weyl-tensor shown in figure 1 one gets a simple graphical representation of C_W^3 and C_M^3 (see figure 2)

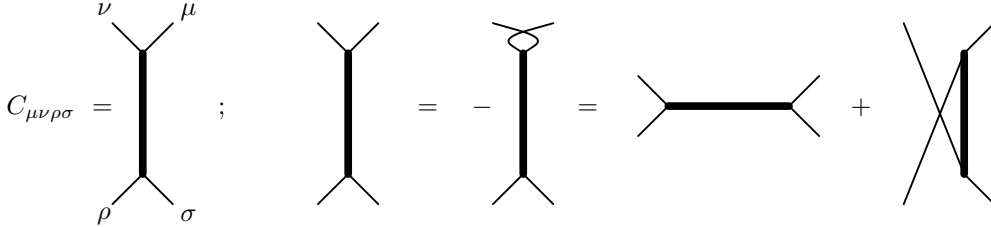


Figure 1: Graphical notation for the Weyl-tensor and its symmetries.

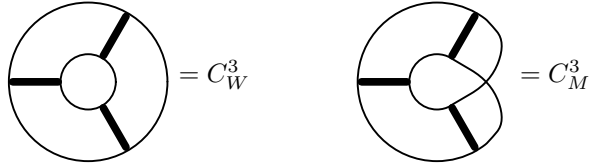


Figure 2: Graphical representation of the Wheel and Möbius graphs

In order to set the stage for our one loop calculation, we now make the usual background-quantum splitting via the replacement $g_{\mu\nu} \rightarrow g_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa^2 = 32\pi G$. The action

S of Eq (1) then splits into $\sum \kappa^{n-2} S^{(n)}$, the superscript n indicating the order in the quantum field h . We find

$$S^{(1)} = - \int dv \sum_{k=0}^{[(d-1)/2]} 2^{-k} (2k+1)! \lambda_k h^{[\mu_1}_{\mu_2 \mu_3} \dots R^{\mu_{2k} \mu_{2k+1}}_{\mu_{2k} \mu_{2k+1}}] \quad (4)$$

Note that this expression is totally antisymmetric on $2k+1$ indices, so on one more index than in Eq (1). The field equation can be read off directly from Eq (3). E.g. in $d=6$

$$\lambda_0 \delta^\mu_\nu + 3 \delta^{[\mu}_\nu R^{\rho\sigma]}_{\rho\sigma} + 30 \lambda_2 \delta^{[\mu}_\nu R^{\rho\sigma}_{\rho\sigma} R^{\tau\omega]}_{\tau\omega} = 0 \quad (5)$$

Upon writing out the indicated antisymmetrizations, one recognizes here the Einstein and Bach-Lanczos tensors in the second and third term, respectively. Observe that the field equation is covariantly conserved: Applying ∇_μ and using the Bianchi identity, each term vanishes separately, due to the invariance under general coordinate transformations. The contracted field equation reads

$$d\lambda_0 + (d-2)R + (d-4)\lambda_2(R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2) = 0 \quad (6)$$

At second order in the quantum fields, we find

$$S^{(2)} = \int dv \sum_{k=0}^{[(d-1)/2]} 2^{-k} (2k+1)! \lambda_k \left[\frac{1}{2} (h^\alpha_{\mu_1} h^{\mu_1}_\alpha - \frac{1}{2} h h^{\mu_1}_{\mu_1}) R^{\mu_2 \mu_3}_{\mu_2 \mu_3} \dots R^{\mu_{2k} \mu_{2k+1}}_{\mu_{2k} \mu_{2k+1}} \right. \\ \left. + k h^{[\mu_1}_{\mu_2 \mu_3} \dots R^{\mu_{2k-2} \mu_{2k-1}}_{\mu_{2k-2} \mu_{2k-1}} (h^{\mu_{2k} \mu_{2k+1}}_{\mu_{2k} \mu_{2k+1}} + \frac{1}{2} R^{\mu_{2k} \mu_{2k+1}}_{\mu_{2k} \mu_{2k+1}} h^\alpha_{\mu_{2k+1}}) \right] \quad (7)$$

where we used semi-colon to denote covariant differentiation. Explicitly in $d=6$

$$S^{(2)} = \int dv \frac{1}{2} [\lambda_0 (h^\mu_\nu h^\nu_\mu - \frac{1}{2} h^2) \\ + 3 ((h^\alpha_\mu h^\mu_\alpha - \frac{1}{2} h h^\mu_\mu) R^{\nu\rho}_{\nu\rho} + 2 h^\mu_\mu h^{\nu;\rho}_{\nu;\rho} + h^\mu_\mu R^{\nu\rho}_{\nu\rho} h^\alpha_\alpha) \\ + 30 \lambda_2 ((h^\alpha_\mu h^\mu_\alpha - \frac{1}{2} h h^\mu_\mu) R^{\nu\rho}_{\nu\rho} R^{\sigma\tau}_{\sigma\tau} + 4 h^\mu_\mu R^{\nu\rho}_{\nu\rho} h^{\sigma;\tau}_{\sigma;\tau} + 2 h^\mu_\mu R^{\nu\rho}_{\nu\rho} R^{\sigma\tau}_{\sigma\tau} h^\alpha_\alpha)] \quad (8)$$

We emphasize that the form of $S^{(2)}$ is completely determined by the requirement of total antisymmetry in its indices; our calculations are only needed to find the numerical coefficients in Eq (8). We also note that the covariant derivatives in the penultimate term of Eq (8) can be moved via partial integration from one field h to another by grace of the Bianchi identity and the indicated total antisymmetry. The last line of Eq (8) shows that in a general curved background the Gauss-Bonnet term does contribute to the quadratic terms and hence to the wave operator. Only in special backgrounds, e.g. in a flat space, does its contribution vanish. Eq (8) can be written as $\int h \Delta h$ where the wave operator takes the schematic form

$$\Delta = \sum_{k=1} \lambda_k R^{k-1} \nabla \nabla + \sum_{k=0} \lambda_k R^k \quad (9)$$

Here, R represents the Riemann tensor or any of its contractions. In the next section we will show that the leading term of the first sum can be gauge fixed to the Laplacian but we will also demonstrate that the Gauss-Bonnet term unavoidably makes the full wave operator non-minimal. In maximally symmetric spaces the wave operator reduces to the minimal form $(1 + c\lambda_2 R)\square$ with R the Ricci scalar and c a number. This fact was exploited in [20]. We will not assume such a special background.

3 Gauge fixing the Einstein-Gauss-Bonnet action

Before we can do quantum calculations, we need to fix the gauge. We choose the usual background covariant harmonic, or DeWitt-Feynman, gauge. We do so by adding the following gauge fixing term to the classical action in Eq (8)

$$S_{fix} = \int dv \, g^{\mu\nu} F_\mu F_\nu \quad , \quad F_\mu = h_{\mu;\nu}^\nu - \frac{1}{2} h_{;\mu} \quad (10)$$

Note that this will gauge fix the Einstein-Hilbert term but not the other terms. This suffices to define the propagator but does not affect the contributions coming from the Gauss-Bonnet term. Hence, we will wind up with a non-minimal wave operator. One might think that a more clever gauge choice would return us to a minimal situation. We have investigated two options:

i) Consider $F_\mu M^{\mu\nu} F_\nu$ with a background dependent but non-differential operator M . This cannot yield a minimal graviton wave operator because M can only be the Ricci, not the Riemann, tensor.

ii) Consider adding[‡] $\zeta R_\mu^{\rho\sigma\nu} h_{\rho\sigma;\nu}$ to F_μ with new gauge parameter ζ . The new terms in F^2 then indeed affect the Gauss-Bonnet contributions to the quantum action. However, we have verified that no choice of the parameter ζ will cure the wave operator and make it minimal.

Thus, we are inevitably led to a non-minimal wave operator. This implies that at quadratic order in the quantum fields ϕ^i , the action takes the form

$$S^{(2)} = \int dv \, \left(\frac{1}{2} P_{ij} \phi^i D^2 \phi^j + \frac{1}{2} X_{ij} \phi^i \phi^j + \frac{1}{2} W_{ij}^{\mu\nu} D_\mu \phi^i D_\nu \phi^j \right) \quad (11)$$

where P , X and W are symmetric matrices which are in general background field dependent. Without loss of generality, we may also assume that $W^{\mu\nu} = W^{\nu\mu}$. The covariant derivatives satisfy

$$[D_\mu, D_\nu]^j{}_k \phi^k = Y_{\mu\nu}{}^j{}_k \phi^k \quad (12)$$

The tensor W is dimensionless and the corresponding vertex can be inserted arbitrarily often into a one loop diagram without changing its degree of divergence. This is the reason why no simple algorithm which generates *all* one loop divergences for non-minimal operators as in Eq (11) is known.

Some partially successful attempts were made though, on which we now comment. Pronin and Stepanyantz [24] relied on the diagrammatic methods of 't Hooft to derive what they call "master formulas for the divergent part of the one loop effective action for arbitrary (both minimal and non-minimal) operators of any order in 4-dimensional curved space". However, the authors of [24] assume that W , called K in their work, is covariantly constant (see the remarks following Eq (28) of [24]). Basically, they assume that the tensor W is always made from products of metric tensors, but this is not the case in the present situation. Avramidi [25] and Avramidi and Branson [16, 17] systematically extended the Schwinger-DeWitt method to non-minimal operators, called non-Laplace type operators there, and diagonal values of the heat kernel coefficients a_0 and a_1 (but not yet a_2) were given[§]. In particular, the so-called commutative limit defined in Eq (4.2)

[‡]Such a gauge was first considered in [21] in the context of Einstein gravity. There, it was erroneously claimed that the choice of ζ could affect the numerical coefficient of the well-known non-renormalizable two loop on shell divergence of this theory. See [22] and [23] for the correct statements.

[§]In [25], $[a_1]$ is proposed as a possible non-abelian generalization of gravity.

of [25] coincides with our approach, but also in [17] it is assumed that W , called a there, is covariantly constant, which as already noted is not the case in the present situation. Already in 1985 Barvinsky and Vilkovisky [13] perturbatively extended the Schwinger-DeWitt method to non-minimal operators at one loop order. Their main application was to operators of the kind $\delta_\beta^\alpha \square - \lambda \nabla^\alpha \nabla_\beta$ for vector fields, where the second term is due to non-minimal gauge fixing. Non-minimal operators for gravitation were also covered. We have a similar situation, but the offending contribution generated by the Gauss-Bonnet term is already of first order in the background curvature. In the parlance of [13], our non-minimal term has background dimensionality $O(1/\ell^2)$. This allows us to treat the W -term as a perturbation which simplifies our analysis a lot[¶]. For a flat background spacetime and to second order in W , an algorithm was first provided Ichinose et al. [14]. This work seems to have gone unnoticed so far. No special properties of W were assumed in [14]. We have checked and confirmed this algorithm and extended it to a curved but Ricci flat space plus some terms cubic in W .

Since we will treat the W -term perturbatively, we choose to define the propagator $G^{kj}(x, x')$ as the inverse of the minimal part of the wave operator

$$(PD^2 + X)_{ik} G^{kj} = -\delta_i^j \delta \quad (13)$$

Comparing $S^{(2)} + S_{\text{fix}}$, Eq (8) plus (10), with Eq (11), we read off that for Ricci flat background metric we have

$$\begin{aligned} P^{\mu\nu\rho\sigma} &= g^{\mu(\rho} g^{\sigma)\nu} - \frac{1}{2} g^{\mu\nu} g^{\rho\sigma} \\ X^{\mu\nu}_{\rho\sigma} &= -\lambda_0 \delta_{\rho\sigma}^{\mu\nu} + 2C^{\mu}_{(\rho}{}^{\nu}_{\sigma)} + \lambda_2 (4U^{\mu\nu}_{(\rho\sigma)} - 2U^{\mu}_{(\rho}{}^{\nu}_{\sigma)} - 2U^{\mu}_{(\rho\sigma)}{}^{\nu}) \\ (Y_{\kappa\lambda})^{\mu\nu}_{\rho\sigma} &= 2C_{\kappa\lambda}^{(\mu}{}^{\nu}_{\rho\sigma)} \\ (W_{\kappa\lambda})^{\mu\nu}_{\rho\sigma} &= 16\lambda_2 \delta^{(\mu}_{(\kappa} C_{\lambda)(\rho}{}^{\nu)}_{\sigma)} - 8\lambda_2 C_{\kappa}^{(\mu}{}^{\nu)}_{\lambda} g_{\rho\sigma} + 8\lambda_2 \delta^{(\mu}_{(\rho} C_{\sigma)(\kappa}{}^{\nu)}_{\lambda)} - 4\lambda_2 g_{\kappa\lambda} C^{\mu}_{(\rho}{}^{\nu}_{\sigma)} \end{aligned} \quad (14)$$

Here, the first two terms in W still need to be symmetrized under pair interchange $\mu\nu \leftrightarrow \rho\sigma$. In fact, W is then totally symmetric under interchanges $\kappa\lambda \leftrightarrow \mu\nu \leftrightarrow \rho\sigma$. In the second expression we defined

$$U_{\mu\nu\rho\sigma} \equiv C_{\mu}{}^{\kappa}{}_{\rho}{}^{\lambda} C_{\nu\kappa\sigma\lambda} \quad (15)$$

Note that after contracting a pair of Weyl tensors twice with each other one can always put the indices in this standard order [26]. Tensor U has the following symmetries

$$U_{\mu\nu\rho\sigma} = U_{\rho\sigma\mu\nu} = U_{\nu\mu\sigma\rho} \quad (16)$$

Furthermore, the field equation allows us to require U to be traceless on any pair of its indices, i.e. we will drop triple or fully contracted pairs of Weyl tensors. We note that

$$(W_{\kappa}{}^{\kappa})^{\mu\nu}_{\rho\sigma} = (W_{\rho\sigma})^{\mu\nu}{}_{\kappa}{}^{\kappa} = 4(4-d)\lambda_2 C^{\mu}_{(\rho}{}^{\nu}_{\sigma)} \quad , \quad \nabla_{\kappa} (W^{\kappa}_{\lambda})^{\mu\nu}_{\rho\sigma} = 0 \quad (17)$$

which will be essential in simplifying our analysis. In $d = 4$, the Gauss-Bonnet term is topological, hence it is invariant under any change of the metric and in particular under conformal transformations. This explains the vanishing of the traces. The vanishing of

[¶]Cf the analysis in sect. 3 of [13] where it is shown that it is better to analyze non-minimal gauge theories via the method of Ward identities. This leads namely to powers of the extremal, i.e. the background field equation.

the divergence of W has its origin in the general coordinate invariance of the Gauss-Bonnet term which was preserved by our choice of gauge.

In principle, we should use the configuration-space metric and its inverse to raise and lower pairs of indices on the various tensors. However, in Ricci flat spaces there is effectively no difference between the various forms of X (this assumes we drop three-fold contractions of two Riemann tensors in the part of X arising from the Gauss-Bonnet term, but such a contraction can anyhow be rewritten via the field equation (5) and does not contribute to the C^3 scalars). The same is true for $Y_{\mu\nu}$. In $d = 4$ it is also true for $W_{\mu\nu}$ because it is traceless there. In particular, the terms in X involving a pair of three-fold contracted Weyl tensors can be dropped.

4 One loop divergences for Einstein-Gauss-Bonnet

To find the one loop divergences for the Einstein-Gauss-Bonnet action, we view it as an action of the form presented in Eq (11) and treat the W -vertex perturbatively. To zeroth order in W and in a Ricci flat space, the divergent part of the one loop effective action for a minimal wave operator as in Eq (13) is known to be given by

$$\begin{aligned} \frac{1}{2} \text{Tr} (\ln G)_{\text{div}} &= \frac{1}{16\pi^2\epsilon} \text{Tr} [a_2] \\ &= \frac{1}{16\pi^2\epsilon} \text{Tr} \left(\frac{1}{2} X^2 + \frac{1}{12} Y^2 + \frac{1}{180} C^2 \right) \end{aligned} \quad (18)$$

Here $\epsilon = 4-d$, $[a_2]$ signifies the diagonal value of the second heat kernel coefficient [27], Tr denotes the functional trace operation, $Y^2 = Y^{\mu\nu} Y_{\mu\nu}$ and $C^2 = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$. Actually, Eq (14) implies that only the X^2 term contributes to C^3 divergences.

To find the first order effect of the W -perturbation, we insert one W -vertex in the one loop graph and determine its divergent part in a Ricci flat space, namely

$$\begin{aligned} \frac{1}{2} \text{Tr} (W^{\mu\nu} D_\mu G \overleftarrow{D}_{\nu'})_{\text{div}} &= \frac{1}{16\pi^2\epsilon} \text{Tr} (W^{\mu\nu} (D_\mu [D_\nu a_1] - [D_\mu D_\nu a_1] + \frac{1}{2} g_{\mu\nu} [a_2]) \\ &= \frac{1}{16\pi^2\epsilon} \text{Tr} \left(\frac{1}{6} W^{\mu\nu} (D_\mu D_\nu X + Y_\mu{}^\rho Y_{\rho\nu} - \frac{1}{15} C_{\mu\rho\sigma\tau} C_\nu{}^{\rho\sigma\tau} \right. \\ &\quad \left. + \frac{1}{4} W^\mu{}_\mu (X^2 + \frac{1}{3} D^2 X + \frac{1}{6} Y^2 + \frac{1}{90} C^2) \right) \end{aligned} \quad (19)$$

Due to Eq (17) and the remarks at the end of section 3 only the $W^{\mu\nu} Y_\mu{}^\rho Y_{\rho\nu}$ term is relevant.

Inserting two W -vertices and again finding the divergent part yields

$$\begin{aligned} &\frac{1}{4} \text{Tr} (W^{\mu\nu}(x) D_\nu G(x, x') \overleftarrow{D}_{\rho'} W^{\rho'\sigma'}(x') D_{\sigma'} G(x', x) \overleftarrow{D}_\mu)_{\text{div}} \\ &= \text{Tr} \left(\frac{1}{2} [W^{\mu\nu} a_1 W^{\rho'\sigma'} D_{\sigma'} a_0 \overleftarrow{D}_\mu] G_{1,\nu\rho'} G_0 + \frac{1}{2} [W^{\mu\nu} a_0 W^{\rho'\sigma'} D_{\sigma'} a_1 \overleftarrow{D}_\mu] G_{0,\nu\rho'} G_1 \right. \\ &\quad + [W^{\mu\nu} a_1 W^{\rho'\sigma'} a_0 \overleftarrow{D}_\mu] G_{1,\nu\rho'} G_{0,\sigma'} + [W^{\mu\nu} a_0 W^{\rho'\sigma'} a_1 \overleftarrow{D}_\mu] G_{0,\nu\rho'} G_{1,\sigma'} \\ &\quad + \frac{1}{2} [W^{\mu\nu} a_0 W^{\rho'\sigma'} a_1] G_{0,\nu\rho'} G_{1,\mu\sigma'} + \frac{1}{2} [W^{\mu\nu} a_0 W^{\rho'\sigma'} a_2] G_{0,\nu\rho'} G_{2,\mu\sigma'} \\ &\quad \left. + \frac{1}{4} [W^{\mu\nu} a_1 W^{\rho'\sigma'} a_1] G_{1,\nu\rho'} G_{1,\mu\sigma'} \right)_{\text{div}} \end{aligned} \quad (20)$$

Here we inserted a heat kernel expansion for each Green function, see [26, 28], and distributed the derivatives, keeping only such terms which in the end can contribute to C^3 . In particular, we dropped $W^\mu{}_\mu$, $D_\mu W^\mu{}_\nu$, $D_\mu X$ and $D_\mu D_\nu X$. A list for the divergent

products appearing here can be found in [26]. We repeat here the few ones that are actually needed (see Eq (D.19) of [26])

$$\begin{aligned}
G_{1,\mu\nu'}G_0 &= G_{0,\mu\nu'}G_1 = G_{0,\mu}G_{1,\nu} = \frac{1}{2}g_{\mu\nu}\delta \\
G_{0,\mu\nu'}G_{1,\rho} &= \frac{1}{6}(g_{\mu\nu}\nabla_\rho - 2g_{\nu\rho}\nabla_\mu - 2g_{\mu\rho}\nabla_\nu)\delta \\
G_{1,\mu\nu'}G_{0,\rho} &= \frac{1}{6}(2g_{\mu\nu}\nabla_\rho - g_{\nu\rho}\nabla_\mu - g_{\mu\rho}\nabla_\nu)\delta \\
G_{0,\mu\nu'}G_{2,\rho\sigma'} &= 2G_{1,\mu\nu'}G_{1,\rho\sigma'} = \frac{1}{2}g_{(\mu\nu}g_{\rho\sigma)}\delta \\
G_{1,\mu\nu'}G_{0,\rho\sigma'} &= \frac{1}{6}C_{\mu(\rho\sigma)\nu}\delta + \nabla\nabla\delta \text{ -terms}
\end{aligned} \tag{21}$$

where we omitted a factor $(16\pi^2\epsilon)^{-1}$ on the far right hand sides. After substituting these expressions, we partially integrate the covariant derivatives off the δ -functions and perform the integration over x' . The final result for the relevant terms is

$$\begin{aligned}
&\frac{1}{4}\text{Tr} (W^{\mu\nu}D_\nu G\overleftarrow{D}_{\rho'}W^{\rho'\sigma'}D_{\sigma'}G\overleftarrow{D}_\mu)_{\text{div}} \\
&= \frac{1}{16\pi^2\epsilon} \text{Tr} \left(\frac{1}{12}W^\mu{}_\nu W^\nu{}_\mu X^2 + \frac{1}{24}W^\mu{}_\nu XW^\nu{}_\mu X + \frac{1}{24}W^\mu{}_\nu D^2W^\nu{}_\mu X \right. \\
&\quad \left. + \frac{1}{4}W^\mu{}_\nu W^\nu{}_\rho XY^\rho{}_\mu + \frac{1}{12}W^\mu{}_\nu XW^\nu{}_\rho Y^\rho{}_\mu + \frac{1}{12}W^\mu{}_\nu W^\rho{}_\sigma XC_{\mu\rho}{}^{\nu\sigma} \right)
\end{aligned} \tag{22}$$

where the right hand side is local. The last term explicitly involves the curvature and is absent from the flat-space algorithm presented in [14]. We have verified and agree with the complete algorithm of [14]; cf their Eq (15). Note that the sign of Y in the present study is opposite to that in [14].

Finally, because X contains a term of zeroth order in the Weyl tensor, see Eq (14), we also need the two W^3X^2 invariants, whose coefficients are easily determined.

Adding the various contributions and keeping only terms which can produce C^3 , our result is

$$\begin{aligned}
\Gamma_{\text{div}}^{(1)} &= \frac{1}{16\pi^2\epsilon} \text{Tr} \left[\frac{1}{2}X^2 + \frac{1}{6}W^\mu{}_\nu Y^\nu{}_\rho Y^\rho{}_\mu + \frac{1}{12}W^\mu{}_\nu W^\nu{}_\mu X^2 + \frac{1}{24}W^\mu{}_\nu XW^\nu{}_\mu X \right. \\
&\quad \left. + \frac{1}{4}W^\mu{}_\nu W^\nu{}_\rho XY^\rho{}_\mu + \frac{1}{12}W^\mu{}_\nu XW^\nu{}_\rho Y^\rho{}_\mu + \frac{1}{24}W^\mu{}_\nu D^2W^\nu{}_\mu X \right. \\
&\quad \left. + \frac{1}{12}W^\mu{}_\nu W^\rho{}_\sigma XC_{\mu\rho}{}^{\nu\sigma} + \frac{1}{24}W^\mu{}_\nu W^\nu{}_\rho W^\rho{}_\mu X^2 + \frac{1}{24}W^\mu{}_\nu W^\nu{}_\rho XW^\rho{}_\mu X \right]
\end{aligned} \tag{23}$$

Calculations with the *Mathematica* [29] package *MathTensor* [30] yield in the graviton sector

$$\begin{aligned}
\text{tr} [X^2] &= -12\lambda_1\lambda_2(C_W^3 - 2C_M^3) \\
\text{tr} [W^\mu{}_\nu Y^\nu{}_\rho Y^\rho{}_\mu] &= -12\lambda_2(C_W^3 - 2C_M^3) \\
\text{tr} [W^\mu{}_\nu W^\nu{}_\mu X^2] = \text{tr} [W^\mu{}_\nu XW^\nu{}_\mu X] &= 576\lambda_0\lambda_1\lambda_2^2(C_W^3 - 2C_M^3) \\
\text{tr} [W^\mu{}_\nu W^\nu{}_\rho XY^\rho{}_\mu] = \text{tr} [W^\mu{}_\nu XW^\nu{}_\rho Y^\rho{}_\mu] &= -72\lambda_1\lambda_2^2(C_W^3 - 2C_M^3) \\
\text{tr} [W^\mu{}_\nu D^2W^\nu{}_\mu X] &= 0 \\
\text{tr} [W^\mu{}_\nu W^\rho{}_\sigma X]C_{\mu\rho}{}^{\nu\sigma} &= 144\lambda_0\lambda_2^2(C_W^3 - 2C_M^3) \\
\text{tr} [W^\mu{}_\nu W^\nu{}_\rho W^\rho{}_\mu X^2] = \text{tr} [W^\mu{}_\nu W^\nu{}_\rho XW^\rho{}_\mu X] &= 2160\lambda_0^2\lambda_2^3(C_W^3 - 2C_M^3)
\end{aligned} \tag{24}$$

Note that each invariant by itself is proportional to E_6 . The abbreviations C_W^3 and C_M^3 were defined in Eq (3) In the here chosen DeWitt-Feynman gauge, the ghost fields do not contribute to the on shell C^3 divergences. Substitution of Eq (14) into Eq (23) produces our final answer

$$\Gamma_{\text{div}}^{(1)} = -\frac{\lambda_2}{16\pi^2\epsilon} \left[2 + 6\lambda_1 + 12\lambda_0\lambda_2 - 72\lambda_0\lambda_1\lambda_2 - 180\lambda_0^2\lambda_2^2 \right] (C_W^3 - 2C_M^3) \tag{25}$$

This is our main result. It shows that the addition of the Gauss-Bonnet term to the sum of the Einstein-Hilbert and cosmological terms induces the on shell six-dimensional Euler density as a new divergence.

One might conjecture that the C^4 divergences would take the form of the dimensionally continued eight-dimensional Euler density E_8 . In a Ricci flat space and using the equation of motion Eq (5), E_8 is proportional to the expression graphically represented in figure 3.

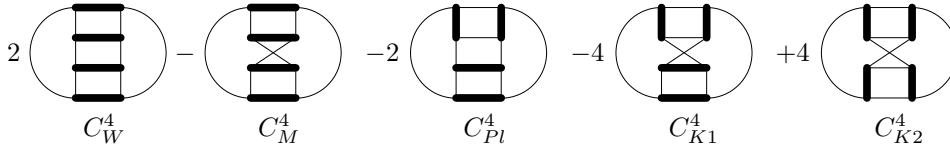


Figure 3: Graphical representation of the Euler density E_8

If we limit our attention to the $\lambda_1^2 \lambda_2^2$ sector we find that only the third and fourth terms in Eq (23) contribute. These two terms yield a divergence proportional to

$$6C_W^4 + 6C_M^4 - 17C_{Pl}^4 - 7C_{K1}^4 + 10C_{K2}^4 \quad . \quad (26)$$

This shows that, in contrast to the E_6 results, the C^4 divergences in the $\lambda_1^2 \lambda_2^2$ sector do not take the form of the Euler density E_8 .

5 Conclusions and discussion

We have demonstrated that the addition of a Gauss-Bonnet term to an action consisting of the Einstein-Hilbert term plus cosmological term induces new on shell divergences at one loop order of the form of the six-dimensional Euler density E_6 . From a strict four-dimensional point of view this implies that the Gauss-Bonnet term has no influence on the one loop renormalizability of gravity. We thus lay to rest this issue, raised long ago by Capper and Kimber [1], at least for the divergent parts of one loop diagrams. The question remains open for finite one loop scattering processes and also for a possible influence of the Gauss-Bonnet term on the two loop divergences of gravity. Still, given the topological nature of the Gauss-Bonnet term in $d = 4$, our result is as expected. *Ex nihilo nihil fit*.

In retrospect we might say that 't Hooft and Veltman [12] showed that if one starts from the Einstein-Hilbert term one induces the Gauss-Bonnet term at one loop. Later, Christensen and Duff [31] added a cosmological constant to the classical action with again the Gauss-Bonnet term being induced. Of course, power counting suffices to predict this, because there is only a single scalar quadratic in the Weyl tensor. This is not anymore so if one includes the Gauss-Bonnet term in the classical action as we did. There then exist two different cubic scalars, but our calculations show that they appear only in the combination corresponding to the six-dimensional Euler density. It leads us to the conclusion that from the d -dimensional point of view advocated by Capper and Kimber [1], one loop renormalizability of Einstein gravity requires one to extend it to a Lovelock theory. However, preliminary calculations show that at quartic order in the Weyl tensor it is not the eight-dimensional Euler density E_8 which appears, but rather several non-topological invariants which vanish in $d = 4$. It is conceivable that one can arrange for such extra terms to produce just E_8 by a special choice of coupling constants. That

would constitute an interesting constraint on the parameters of the class of all Lovelock theories. So far, such constraints were based on stability [32, 20] or causality [33]; see also [34]. We hope to return to this issue in future.

Our work can also be seen as an interesting setting in which non-minimal wave operators occur. We have verified the one loop algorithm of [14] for such operators and extended it to curved Ricci flat spaces, both via 't Hooft's non-covariant method and by using the covariant Schwinger-DeWitt method. Our result does not depend on the precise coefficients in the algorithm: Every contribution individually reduces to the six-dimensional Euler density.

Possibly our work can be continued to all orders in the non-minimal term by extending the methods of [13, 16, 17, 24] to the case of divergenceless rather than covariantly constant W -tensor.

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